Practica 13

Integrales Triples

Problema 1.

Dados los siguientes campos vectoriales, compruebe la validez del teorema de Green en el plano, considerando el dominio de integración la región:

$$R : x^2 + y^2 \leq a^2$$

Circulando sobre la curva:

$$C : \mathbf{r}(t) = (a\cos(t))\mathbf{i} + (a\mathrm{sen}(t))\mathbf{j}, 0 \leq t \leq 2\pi$$

1. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

2. $\mathbf{F} = y\mathbf{i}$

3. $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$

4. $\mathbf{F} = -x^2y\mathbf{i} + xy^2\mathbf{j}$

Ejercicio 1

Recordemos que para un campo vectorial

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j}$$

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

Flujo saliente

Integral de la divergencia

Recordemos que para un campo vectorial

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j}$$

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Circulación anti-horaria

Integral del Rotacional

1)

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx$$

$$\mathrm{M = -y = -a\ sin\ t,\ N = x = a\ cos\ t,}$$

$$\mathrm{dx = -a\ sin\ t\ dt,\ dy = a\ cos\ t\ dt}$$

$$\oint_{\mathrm{C}} \mathrm{M\ dy - N\ dx} = \int_0^{2\pi} \mathrm{[(-a\ sin\ t)(a\ cos\ t) - (a\ cos\ t)(-a\ sin\ t)]\ dt} =$$

$$= \int_0^{2\pi} 0\ \mathrm{dt} = 0$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$$

$$\frac{\partial \mathrm{M}}{\partial \mathrm{x}} = 0,\ \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = -1,\ \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = 1,\ \frac{\partial \mathrm{N}}{\partial \mathrm{y}} = 0;$$

$$\int\int_{\mathrm{R}} \left(\frac{\partial \mathrm{M}}{\partial \mathrm{x}} + \frac{\partial \mathrm{N}}{\partial \mathrm{y}}\right) \mathrm{dx\, dy} = \int\int_{\mathrm{R}} 0\ \mathrm{dx\, dy} = 0.$$

3)

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx$$

$$\mathrm{M} = -\mathrm{x}^2\mathrm{y} = -\mathrm{a}^3\cos^2\mathrm{t},\ \mathrm{N} = \mathrm{xy}^2 = \mathrm{a}^3\cos\mathrm{t}\sin^2\mathrm{t},$$

$$\mathrm{dx} = -\mathrm{a}\sin\mathrm{t}\,\mathrm{dt},\ \mathrm{dy} = \mathrm{a}\cos\mathrm{t}\,\mathrm{dt}$$

$$\oint_{\mathrm{C}} \mathrm{M\,dy} - \mathrm{N\,dx} = \int_0^{2\pi} (-\mathrm{a}^4\cos^3\mathrm{t}\sin\mathrm{t} + \mathrm{a}^4\cos\mathrm{t}\sin^3\mathrm{t})$$

$$= \left[\tfrac{\mathrm{a}^4}{4}\cos^4\mathrm{t} + \tfrac{\mathrm{a}^4}{4}\sin^4\mathrm{t}\right]_0^{2\pi} = 0$$

$$\iint\limits_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy$$

$$\frac{\partial \mathrm{M}}{\partial \mathrm{x}} = -2\mathrm{xy}, \ \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = -\mathrm{x}^2, \ \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = \mathrm{y}^2, \quad \frac{\partial \mathrm{N}}{\partial \mathrm{y}} = 2\mathrm{xy};$$

$$\int\!\!\int_{\mathrm{R}} \left(\frac{\partial \mathrm{M}}{\partial \mathrm{x}} + \frac{\partial \mathrm{N}}{\partial \mathrm{y}}\right) \mathrm{dx\,dy} = \int\!\!\int_{\mathrm{R}} (-2\mathrm{xy} + 2\mathrm{xy})\,\mathrm{dx\,dy} = 0.$$

Problema 2.

En los siguientes casos calcule la circulación antihoraria, así como el flujo saliente, para los campos vectoriales y curvas dadas.

1) $\mathbf{F}(x,y) = (x-y)\mathbf{i} + (y-x)\mathbf{j}$

C: el cuadrado delimitado por $x=0,\ x=1,\ y=0,\ y=1$

2) $\mathbf{F}(x,y) = (y^2-x^2)\mathbf{i} + (x^2-y^2)\mathbf{j}$

C: el triángulo formado por $y=0,\ x=3,\ y=x$

2) $\mathbf{F}(x,y) = \left(\operatorname{tg}^{-1}\left(\frac{y}{x}\right)\right)\mathbf{i} + \ln(x^2-y^2)\mathbf{j}$

C: La región definida en coordenadas polares por:

$$1 \leq r \leq 2,\ 0 \leq \theta \leq \pi$$

$$\mathbf{F}(x,y) = (x-y)\mathbf{i} + (y-x)\mathbf{j}$$

$$\mathrm{M} = \mathrm{x} - \mathrm{y},\ \mathrm{N} = \mathrm{y} - \mathrm{x} \ \Rightarrow \ \frac{\partial \mathrm{M}}{\partial \mathrm{x}} = 1,\ \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = -1,\ \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = -1,\ \frac{\partial \mathrm{N}}{\partial \mathrm{y}} = 1$$

$$\iint_{\mathrm{R}} 2 \, \mathrm{dx} \, \mathrm{dy} = \int_0^1 \int_0^1 2 \, \mathrm{dx} \, \mathrm{dy} = 2$$

Integral de flujo

$$\iint_{\mathrm{R}} [-1 - (-1)] \, \mathrm{dx} \, \mathrm{dy} = 0$$

Circulación

$$\mathbf{F}(x,y) = (y^2 - x^2)\mathbf{i} + (x^2 - y^2)\mathbf{j}$$

$$\mathrm{M} = \mathrm{y}^2 - \mathrm{x}^2,\ \mathrm{N} = \mathrm{x}^2 + \mathrm{y}^2 \ \Rightarrow\ \frac{\partial \mathrm{M}}{\partial \mathrm{x}} = -2\mathrm{x},\ \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = 2\mathrm{y},\ \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = 2\mathrm{x},\ \frac{\partial \mathrm{N}}{\partial \mathrm{y}} = 2\mathrm{y}$$

$$\iint_{\mathrm{R}} (-2\mathrm{x} + 2\mathrm{y})\,\mathrm{dx}\,\mathrm{dy}$$

Integral de flujo

$$= \int_0^3 \int_0^{\mathrm{x}} (-2\mathrm{x} + 2\mathrm{y})\,\mathrm{dy}\,\mathrm{dx} = \int_0^3 (-2\mathrm{x}^2 + \mathrm{x}^2)\,\mathrm{dx} = \left[-\tfrac{1}{3}\,\mathrm{x}^3\right]_0^3 = -9$$

$$\iint_{\mathrm{R}} (2\mathrm{x} - 2\mathrm{y})\,\mathrm{dx}\,\mathrm{dy}$$

Circulación

$$= \int_0^3 \int_0^{\mathrm{x}} (2\mathrm{x} - 2\mathrm{y})\,\mathrm{dy}\,\mathrm{dx} = \int_0^3 \mathrm{x}^2\,\mathrm{dx} = 9$$

$$\mathbf{F}(x,y) = \left(\operatorname{tg}^{-1}\left(\frac{y}{x}\right)\right)\mathbf{i} + \ln(x^2 - y^2)\mathbf{j}$$

$$\mathrm{M} = \tan^{-1}\frac{\mathrm{y}}{\mathrm{x}}, \mathrm{N} = \ln\left(\mathrm{x}^2 + \mathrm{y}^2\right) \Rightarrow \frac{\partial \mathrm{M}}{\partial \mathrm{x}} = \frac{-\mathrm{y}}{\mathrm{x}^2+\mathrm{y}^2}, \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = \frac{\mathrm{x}}{\mathrm{x}^2+\mathrm{y}^2}, \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = \frac{2\mathrm{x}}{\mathrm{x}^2+\mathrm{y}^2}, \frac{\partial \mathrm{N}}{\partial \mathrm{y}} = \frac{2\mathrm{y}}{\mathrm{x}^2+\mathrm{y}^2}$$

Integral de flujo

$$\iint_{\mathrm{R}}\left(\frac{-\mathrm{y}}{\mathrm{x}^2+\mathrm{y}^2} + \frac{2\mathrm{y}}{\mathrm{x}^2+\mathrm{y}^2}\right)\mathrm{dx}\,\mathrm{dy} = \int_0^{\pi}\int_1^{2}\left(\frac{\mathrm{r}\sin\theta}{\mathrm{r}^2}\right)\mathrm{r}\,\mathrm{dr}\,\mathrm{d}\theta = \int_0^{\pi}\sin\theta\,\mathrm{d}\theta = 2$$

Circulación

$$\iint_{\mathrm{R}}\left(\frac{2\mathrm{x}}{\mathrm{x}^2+\mathrm{y}^2} - \frac{\mathrm{x}}{\mathrm{x}^2+\mathrm{y}^2}\right)\mathrm{dx}\,\mathrm{dy} = \int_0^{\pi}\int_1^{2}\left(\frac{\mathrm{r}\cos\theta}{\mathrm{r}^2}\right)\mathrm{r}\,\mathrm{dr}\,\mathrm{d}\theta = \int_0^{\pi}\cos\theta\,\mathrm{d}\theta = 0$$

Problema 3.

En los siguientes casos calcule el trabajo hecho por la fuerza externa F para mover una partícula una vez en dirección antihorario sobre la curva dada.

1) $\mathbf{F}(x, y) = \left(2xy^3\right)\mathbf{i} + \left(4x^2y^2\right)\mathbf{j}$

El triángulo definido en el primer cuadrante por el eje x y las curvas $x = 1,\ y = x^3$

2) $\mathbf{F}(x, y) = (4x - 2y)\,\mathbf{i} + (2x - 4y)\,\mathbf{j}$

El circulo $(x-2)^2 + (y-2)^2 = 4$

$$\mathbf{F}(x,y) = \left(2xy^3\right)\mathbf{i} + \left(4x^2y^2\right)\mathbf{j}$$

$$\mathrm{M} = 2\mathrm{xy}^3,\ \mathrm{N} = 4\mathrm{x}^2\mathrm{y}^2 \ \Rightarrow\ \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = 6\mathrm{xy}^2,\ \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = 8\mathrm{xy}^2$$

$$\oint_C 2\mathrm{xy}^3\,\mathrm{dx} + 4\mathrm{x}^2\mathrm{y}^2\,\mathrm{dy} = \iint_R \left(8\mathrm{xy}^2 - 6\mathrm{xy}^2\right)\mathrm{dx}\,\mathrm{dy}$$

$$= \int_0^1 \int_0^{\mathrm{x}^3} 2\mathrm{xy}^2\,\mathrm{dy}\,\mathrm{dx} = \int_0^1 \frac{2}{3}\,\mathrm{x}^{10}\,\mathrm{dx} = \frac{2}{33}$$

$$\mathbf{F}(x,y) = (4x - 2y)\,\mathbf{i} + (2x - 4y)\,\mathbf{j}$$

$$\mathrm{M} = 4\mathrm{x} - 2\mathrm{y},\ \mathrm{N} = 2\mathrm{x} - 4\mathrm{y} \ \Rightarrow \ \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = -2,\ \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = 2$$

$$\oint_{\mathrm{C}} (4\mathrm{x} - 2\mathrm{y})\,\mathrm{dx} + (2\mathrm{x} - 4\mathrm{y})\,\mathrm{dy} = \int\!\!\int_{\mathrm{R}} [2 - (-2)]\,\mathrm{dx}\,\mathrm{dy}$$

$$= 4 \int\!\!\int_{\mathrm{R}} \mathrm{dx}\,\mathrm{dy} = 4$$ (área del circulo)

$$= 4(\pi \cdot 4) \boxed{= 16\pi}$$

Problema 4.

Evaluar las siguientes integrales:

1) $$\oint_C (y^2\,dx + x^2\,dy)$$

C Triángulo definido por $x = 0, x + y = 1, y = 0$

2) $$\oint_C (3y\,dx + 2x\,dy)$$

C la frontera de $0 \le x \le \pi, 0 \le y \le \sin x$

$$\oint_C (y^2\,dx + x^2\,dy)$$

$$M = y^2,\ N = x^2 \ \Rightarrow\ \frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 2x$$

$$\oint_C y^2\,dx + x^2\,dy = \iint_R (2x - 2y)\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx = \int_0^1 (-3x^2 + 4x - 1)\,dx$$

$$= \left[-x^3 + 2x^2 - x\right]_0^1 = -1 + 2 - 1 = 0$$

$$\oint_C (3y\,dx + 2x\,dy)$$

$$M = 3y,\ N = 2x \ \Rightarrow \ \frac{\partial M}{\partial y} = 3,\ \frac{\partial N}{\partial x} = 2$$

$$\oint_C 3y\,dx + 2x\,dy = \iint_R (2-3)\,dx\,dy = \int_0^{\pi}\int_0^{\sin x} -1\,dy\,dx$$

$$= -\int_0^{\pi} \sin x\,dx \boxed{= -2}$$

Ejercicio 1.

Deducir la formula de área utilizando el teorema de Green

Utilice el teorema de Green para calcular el área de la región encerrada por las siguientes curvas

a) $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 \leq t \leq 2\pi$

b) $\mathbf{r}(t) = t^2\mathbf{i} + ((t^3/3) - t)\mathbf{j}, \quad -\sqrt{3} \leq t \leq \sqrt{3}$

Ejercicio 2.

(R) a) $\frac{3}{8}\pi$

b) $\frac{8}{5}\sqrt{3}$

Problema 4.

Entre todas las curvas simples cerradas en el plano, orientadas de manera antihoraria, encuentre aquella a lo largo de la cual el trabajo hecho por el siguiente campo de fuerzas es máximo

$$\mathbf{F} = \left(\frac{1}{4}x^2y + \frac{1}{3}y^3\right)\mathbf{i} + x\mathbf{j}$$

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$$\mathbf{F} = \left(\frac{1}{4}x^2y + \frac{1}{3}y^3\right)\mathbf{i} + x\mathbf{j}$$

$$\mathrm{M} = \tfrac{1}{4}\,\mathrm{x}^2\mathrm{y} + \tfrac{1}{3}\,\mathrm{y}^3,\ \mathrm{N} = \mathrm{x} \ \Rightarrow\ \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = \tfrac{1}{4}\,\mathrm{x}^2 + \mathrm{y}^2,\ \frac{\partial \mathrm{N}}{\partial \mathrm{x}} = 1$$

$$\frac{\partial \mathrm{N}}{\partial \mathrm{x}} - \frac{\partial \mathrm{M}}{\partial \mathrm{y}} = 1 - \left(\tfrac{1}{4}\,\mathrm{x}^2 + \mathrm{y}^2\right) > 0$$

Rotacional

$$\boxed{\tfrac{1}{4}\,\mathrm{x}^2 + \mathrm{y}^2 = 1}$$

Region

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(1 - \tfrac{1}{4}\,\mathrm{x}^2 - \mathrm{y}^2\right) \mathrm{dx}\,\mathrm{dy}$$

Trabajo

La integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(1 - \frac{1}{4}\,x^2 - y^2\right) dx\,dy$$

En la región

$$\frac{1}{4}\,x^2 + y^2 = 1$$

Donde

$$1 - \left(\frac{1}{4}\,x^2 + y^2\right) > 0$$

Tendrá valor máximo